Hölder estimates for the Hausdorff distance and a quasi-metric

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Acknowledgements

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2. $\mathcal{K}^n$ is the family of compact, convex sets in $\mathbb{R}^n$.
3. $\mathcal{K}^n$ is a complete metric space with the **Hausdorff distance**

$$d_{\mathcal{H}}(K, L) := \max \left\{ \sup_{x \in K} d(x, L), \sup_{y \in L} d(y, K) \right\}$$

The Minkowski Sum

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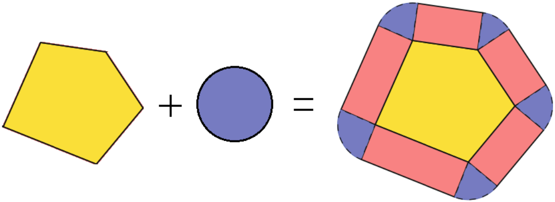

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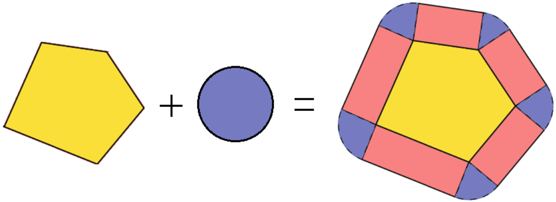

2. If $t \geq 0$, the Minkowski dilate tK is the set

$$tK := \{tx : x \in K\}.$$

Mixed Volumes

1. (Brunn-Minkowski) There are real numbers $MV_j(K, L) \geq 0$ for $0 \leq j \leq n$ such that

$$\mathrm{Vol}(K + tL) = \sum_{j=0}^{n} \binom{n}{j} MV_j(K, L) t^j$$

for all $t \geq 0$.

2. The quantity $MV_j(K, L)$ is called the j-th *mixed volume* of K and L.

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2. We can define a function d_G on $\mathcal{K}(G) \times \mathcal{K}(G)$ by

$$d_G(K, L) = \sum_{j=1}^{n} 2MV_j(G, K \tilde{\cup} L) - MV_j(G, K) - MV_j(G, L).$$

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3. d_G satisfies all the axioms of metric except for the triangle inequality
4. This quantity d_G is the principle object of our study

A Brief Detour into Complex Geometry

1. Darvas-Di Nezza-Lu in [2] define a quasi-metric d between singularity types of psh functions on a Kähler manifold.
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2. d satisfies all the axioms of a metric (i.e., symmetry, non-negativity, and non-degeneracy), except for the triangle inequality.
3. Rather, the quasi-triangle inequality holds:

$$d([u],[v]) \leq D(d([u],[w]) + d([w],[v]))$$

for all singularity types $[u],[v],[w]$ for a fixed $D > 1$.

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$$d([u],[v]) = \sum_{j=1}^{n} 2MV_j(P, K\tilde{\cup}L) - MV_j(P,K) - MV_j(P,L) = d_P(K,L)$$

if K, L are the convex sets associated to singularity types $[u],[v]$.

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   if K, L are the convex sets associated to singularity types $[u],[v]$.
3. d_G is a natural generalization of d_P!

Questions:

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2. Are there inequalities relating d_G and $d_{\mathcal{H}}$?
3. Does the choice of G play any significant role in the definition of d_G?

Our Work

Theorem

There exists $C > 0$ such that

$$d_G(K, L) \leq C \cdot d_{\mathcal{H}}(K, L)$$

for all $K, L \in \mathcal{K}(G)$. Moreover, if $\beta > 1$, then no inequality of the form $d_G(K, L) \leq C \cdot d_{\mathcal{H}}(K, L)^\beta$ can hold for all K, L.

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Improvements in Special Cases

Theorem

If G is a convex polytope, then exists $A > 0$ such that

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Theorem

If G has C^2 boundary, then there exists $A > 0$ such that

$$A \cdot d_{\mathcal{H}}(K, L)^{\frac{n+1}{2}} \leq d_G(K, L)$$

for all $K, L \in \mathcal{K}(G)$.

A First Lipschitz Estimate

We prove the following inequality:

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Idea

1. Mixed volumes are Lipschitz continuous:

$$MV_j(G, K \tilde{\cup} L) - MV_j(G, K) \leq jMV_1(G, B)d_{\mathcal{H}}(K, L)$$

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3. Add everything up:

$$\begin{aligned} d_G(K, L) &= \sum_{j=1}^{n} 2MV_j(G, K \tilde{\cup} L) - MV_j(G, K) - MV_j(G, L) \\ &\leq \sum_{j=0}^{n} 2jMV_1(G, B)d_{\mathcal{H}}(K, L) \\ &\leq 2n^2 MV_1(G, B)d_{\mathcal{H}}(K, L) \end{aligned}$$

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4. $d_G(K, L) \leq C \cdot d_{\mathcal{H}}(K, L)$, where $C = 2n^2 MV_1(G, B)$

Lower Hölder Bound of $d_{\mathcal{H}}$ on d_G

Theorem

There exists $D > 0$ such that

$$D \cdot d_{\mathcal{H}}(K, L)^n \leq d_G(K, L)$$

for all $K, L \in \mathcal{K}(G)$.

Getting Rid of Mixed Volumes

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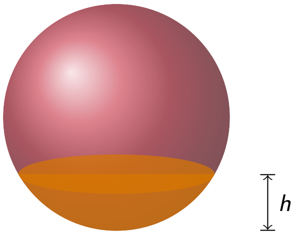
3. So, we can use ρ_G to prove estimates for d_G.

Convex Caps

- A **convex cap** of G is

$$C_G(u,h) := \left\{ x \in G : u\cdot x \geq \sup_{y\in G} u\cdot y - h \right\},$$

for some unit vector u and height $h \geq 0$.



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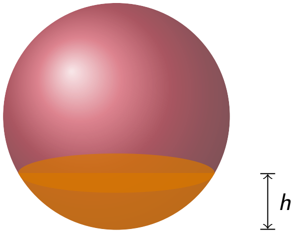

- These convex caps prove to be powerful tool to analyze lower bounds of the form

  $$A \cdot d_{\mathcal{H}}(K,L)^p \leq d_G(K,L)$$

  for general convex bodies G.

Estimation in Four Short Steps

Using caps and getting rid of mixed volumes allows us a new idea to find a lower bound:

1. Need a bound of the form

   $$C \cdot d_{\mathcal{H}}(K,L)^n \leq \rho_G(K,L) = 2V(G + K\tilde{\cup}L) - V(G+K) - V(G+L);$$

   by monotonicity then suffices to get a lower bound by $C \cdot d_{\mathcal{H}}(K,L)^n$ of

   $$V(G + K\tilde{\cup}L) - V(G+K) \quad \text{or} \quad V(G + K\tilde{\cup}L) - V(G+L),$$

   which given the respective containments reduces to bounding volumes of

   $$(G + K\tilde{\cup}L) \setminus (G+K) \quad \text{or} \quad (G + K\tilde{\cup}L) \setminus (G+L).$$

Estimation in Four Short Steps

2. Use some convex geometry to stick a cap of G with height $d := d_{\mathcal{H}}(K, L)$ in one of these sets.
3. Stick a cone in that cap, which has volume itself that looks like d^n up to a constant.
4. Profit!

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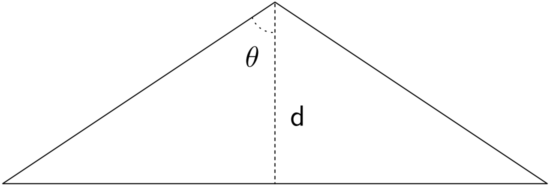

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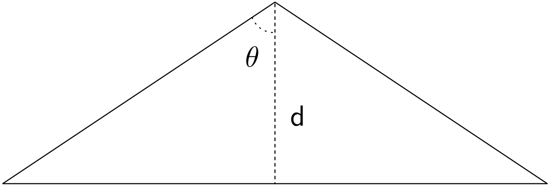

where our cone in this case is our cap, and there is a trigonometric relation relating the length of the base directly to d, such that the area of the cone is d^2 up to a constant.

What's Believable, What's Not?

This idea clearly also gives intuition in higher dimensions; a short argument proves the desired statement entirely. We want then to more more rigorously justify (2).

To do so, we introduce a key lemma that explains why (2) even holds.

A Return to Convex Geometry

Define $(p, q] := (p \tilde{\cup} q) \setminus p$.

Lemma

Let $K, L \in \mathcal{K}^n$. Up to relabeling, there exists $p \in K$ and $q \in L$ such that $||q - p|| = d_{\mathcal{H}}(K, L)$ and $(p, q] \subseteq K \tilde{\cup} L \setminus K$. Moreover, there exists a hyperplane orthogonal to $[p, q]$ which supports K.

A Return to Convex Geometry

Proof:

1. Obtain p, q the natural way by a compactness argument. In particular, take WLOG

$$d_{\mathcal{H}}(K, L) = \sup_{x \in L} d(x, K) = d(p, K) = d(q, K) = |p - q|$$

   for $p \in K, q \in L$.

2. Clearly $(p, q] \subseteq K\tilde{\cup}L-$, so assume $(p, q] \not\subseteq K$; i.e. $\exists t \in (0, 1]$ such that $tp + (1-t)q \in K$. Then

$$|q - ((1-t)p + tq)| = (1-t)|p - q| < |p - q|,$$

   but this is a contradiction, as p is taken as the closest point in K to q.

3. One verifies we also have the supporting hyperplane we want quickly.

A Return to Convex Geometry

Lemma

Let $K, L \in \mathcal{K}(G)$. Then there exists, up to relabeling, $u \in S^{n-1}$ and $q \in L$ such that

$$\operatorname{int}\left(C_G(u, d_{\mathcal{H}}(K, L)) + q\right) \subseteq (G + K\tilde{\cup}L) \setminus (G + K)$$

Proof:

1. Simply let $d := d_{\mathcal{H}}(K, L)$ and take $|q - p| = d$ as done previously. The choice $u = \frac{q-p}{d}$ suffices.

Lower Hölder Bound of $d_{\mathcal{H}}$ on d_G

We've verified every ingredient, so we recover:

Theorem

There exists $D > 0$ such that

$$D \cdot d_{\mathcal{H}}(K, L)^n \leq d_G(K, L)$$

for all $K, L \in \mathcal{K}(G)$.

Optimal Hölder Exponent for Polytopes

Theorem

If G is a convex polytope, then there exists $A > 0$ such that

$$Ad_{\mathcal{H}}(K, L)^n \leq d_G(K, L)$$

holds for all $K, L \in \mathcal{K}(G)$, and no such bound can exist with exponent less than n.

1. For a polytope, a cone asymptotically approximates the volume of a cap. So a cone is the best way to approximate a polygonal cap.
2. So d_G and $d_{\mathcal{H}}^n$ are asymptotically the same for small h, implying the minimality of n.

Improvement for C^2 Bodies

Theorem

If G is a C^2 body, then there exists $A > 0$ such that

$$A \cdot d_{\mathcal{H}}(K, L)^{\frac{n+1}{2}} \leq d_G(K, L)$$

for all $K, L \in \mathcal{K}(G)$.

1. Bound $\mathrm{Vol}(C_G(u, h))$ below by volume of a paraboloid.
2. The volume of the parabaloid behaves like $h^{(n+1)/2}$.

C^2 isn't actually the condition

Rather, it's that our body G supports, tangentially, an ball of constant ϵ inside it; i.e. you can rool an ϵ-ball smoothly inside of it, against its interior.

Works Cited

T. Darvas, E. Di Nezza and C. H. Lu, Log-concavity of volume and complex Monge-Ampere equations with prescribed singularity, Math. Ann. 379 (2021), no. 1-2, 95-132. arXiv:1807.00276.

T. Darvas, E. Di Nezza and C. H. Lu, The metric geometry of singularity types, J. Reine Angew. Math. 771 (2021), 137-170. arXiv:1909.00839

Future Directions:

1. Is d_G a quasi-metric? I.e., does it satisfy the quasi-triangle inequality?
2. Is d_G quasi-isometric to a true metric? I.e., is there a true metric d with
$$c_1 d \leq d_G \leq c_2 d?$$
    1. Partial progress: if $d(K, L) = \mathrm{Vol}((G+K)\triangle(G+L))$, we conjecture d is quasi-isometric to d_G.

Questions?